Cost Minimization

1,3,6

The problem:

- firms operate in BOTH inputs + final good market ($ ↔ $)
  - ↳ they have target production (isoquant) & want to complete most efficiently (min cost!)

Firm problem:

$$\min_{L,K} \overline{Cost} = wL + rK \quad \text{s.t. } f(L,K) = q_0$$

- w: wage rate
- r: rental rate of capital
- q_0: goal output level

w & r are given

[min cost to produce set output level]

isocost vs isoquant

- isocost: combinations of L & K that cost the same
- isoquant: combinations of L & K that produce same output

$wL + rK = C$

$L = 0 \Rightarrow rK = c$

$K = \frac{c}{r}$

$K = 0 \Rightarrow wL = c$

$L = \frac{c}{w}$

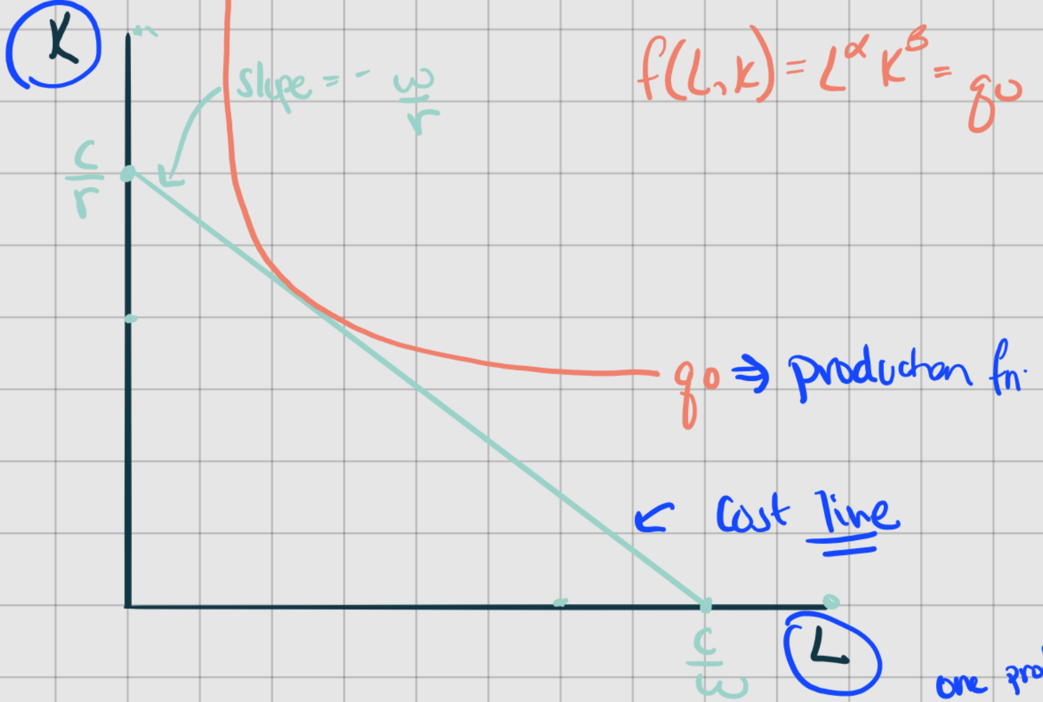
slope $\Rightarrow wL + rK = c$

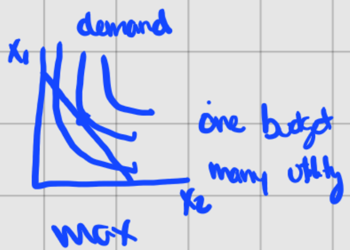
$rK = c - wL$

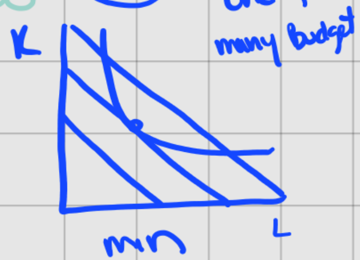
$K = \frac{c}{r} - \frac{w}{r}L$ ← input price ratio

$y = b - mx$

$f(L,K) = L^{\alpha}K^{\beta} = q_0$

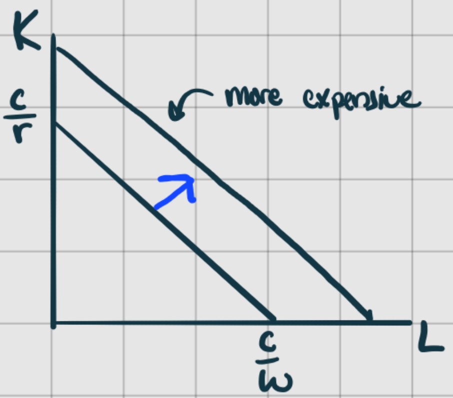




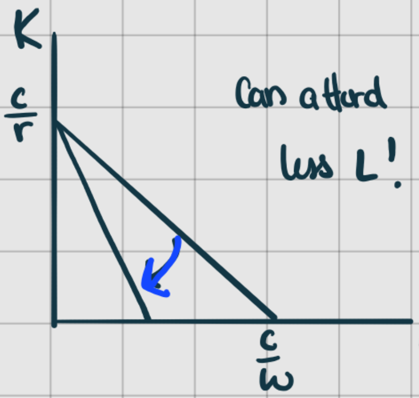


Shifts in isocost:

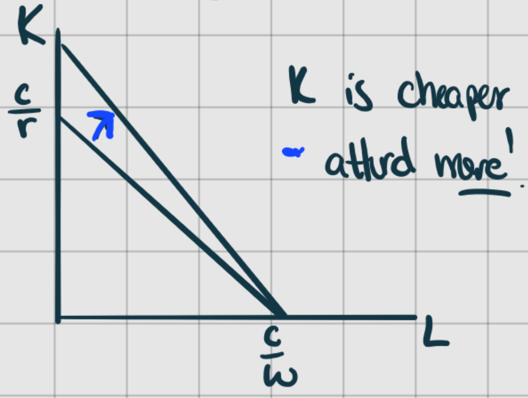
↑c



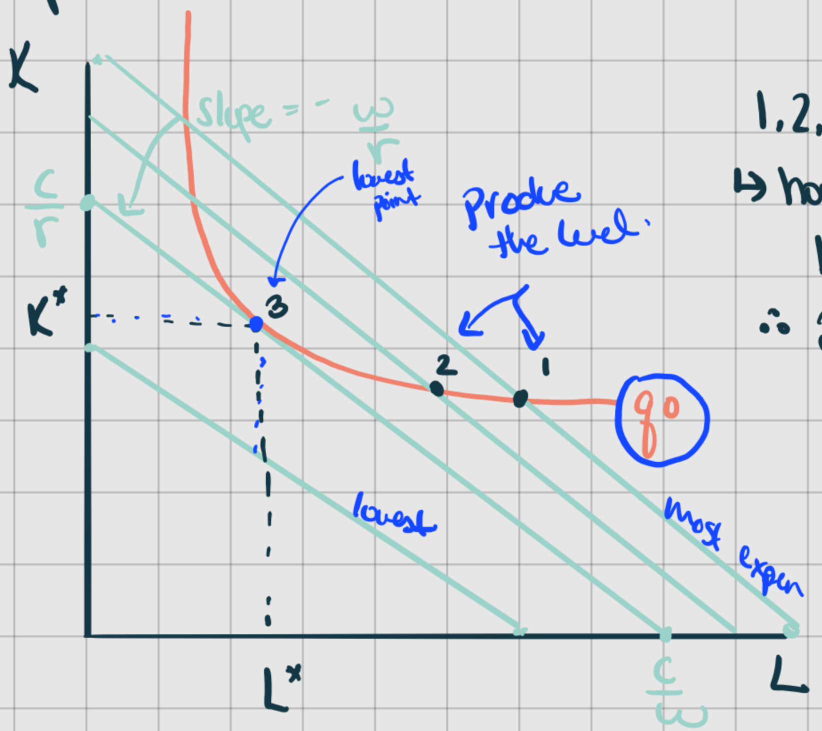
↑w



↓r



How to find optimal:

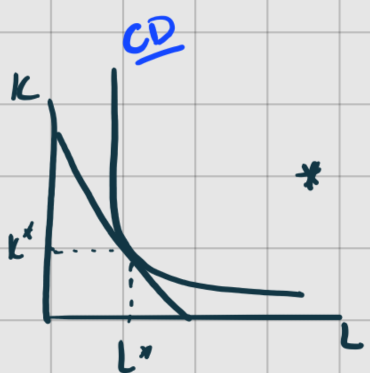


1,2,3 produce same q.
↳ however cost q
1 > 2 > 3
∴ 3 produce q for lowest cost!

1) Tangency solution

$$\min \; wL + rK$$

changes → s.t. $f(L,K) = q = L^{\alpha} K^{\alpha}$



* Slope of isocost = Slope of isoquant *

$TRS = -\frac{w}{r}$ ← input price ratio

* tangency / optimality condition *

$\frac{-MP_L}{MP_K} = -\frac{\partial f(L,K)/\partial L}{\partial f(L,K)/\partial K}$

ex: $f(L,K) = \boxed{L \cdot K}$

(1) tangency: $\frac{K}{L} = \frac{w}{r}$ (MPL → K, MPK → L) $K = \frac{w}{r} \cdot L$

sub into ↓

(2) constraint: $L \cdot K = q$ ← production function.

$L \cdot \left(\frac{wL}{r}\right) = q \quad L^2 \frac{w}{r} = q$

$L^* = \sqrt{q \frac{r}{w}}$ (sub into)

$L^*\ K^*$ → find in terms of q, r, w → given

$r^{-1+\frac{1}{2}} = r^{-\frac{1}{2}}$

$w^{1-\frac{1}{2}} = w^{\frac{1}{2}}$

$K = \frac{wL}{r} = \frac{w}{r}\sqrt{q\frac{r}{w}} \Rightarrow K^* = \sqrt{q\frac{w}{r}}$

$C = wL + rK$

∴ cost function is a function of q, r, w

Sol^n needs [$L^* + K^*$ → - cost function]

$c(q,r,w) = wL^*(q,r,w) + rK^*(q,r,w)$

$= w\sqrt{q\frac{r}{w}} + r\sqrt{q\frac{w}{r}}$

$= \sqrt{qrw} + \sqrt{qwr}$

$= \boxed{2\sqrt{wrq}}$ ⇒ Cost minimization for $f(L,K) = L \cdot K = q$.

Lagrangian: (alternative way to solve!)

$\min \quad C = \underbrace{wL + Kr}_{\text{object}} - \lambda\underbrace{(LK - q)}_{\text{constraint}}$

Tangency condition → $\frac{K}{L} = \frac{w}{r}$

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda K = 0 \Rightarrow \boxed{\frac{w}{K} = \lambda}$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda L = 0 \Rightarrow \boxed{\frac{r}{L} = \lambda}$$

$$\boxed{\frac{w}{K} = \frac{r}{L}}$$

$$K = \frac{wL}{r}$$

sub.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = LK - q = 0 \Rightarrow LK = q$$

$$L\left(\frac{wL}{r}\right) = q$$

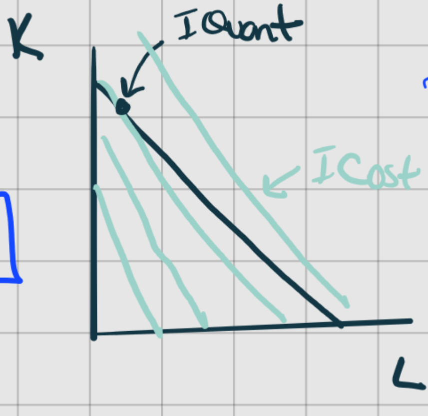
sub

$$L^* = \sqrt{\frac{qr}{w}} \qquad K^* = \sqrt{\frac{qw}{r}}$$

cost function $= wL^* + rK^*$

2) Corner sol^n

$$\boxed{f(L,K) = aL + bK = q}$$



Perfect sub

$$\min \; wL + rK$$
$$\text{st} \quad f(L,K) = aL + bK = q$$

↳ Compare $|TRS|$ vs $\boxed{\frac{w}{r}}$

* what is the better deal? *

$$|TRS| = \left|\frac{MP_L}{MP_K}\right| = \frac{a}{b}$$

$$\frac{MP_L}{MP_K} \text{ vs } \frac{w}{r} \qquad \boxed{\frac{MP_L}{w}} \text{ vs } \boxed{\frac{MP_K}{r}}$$

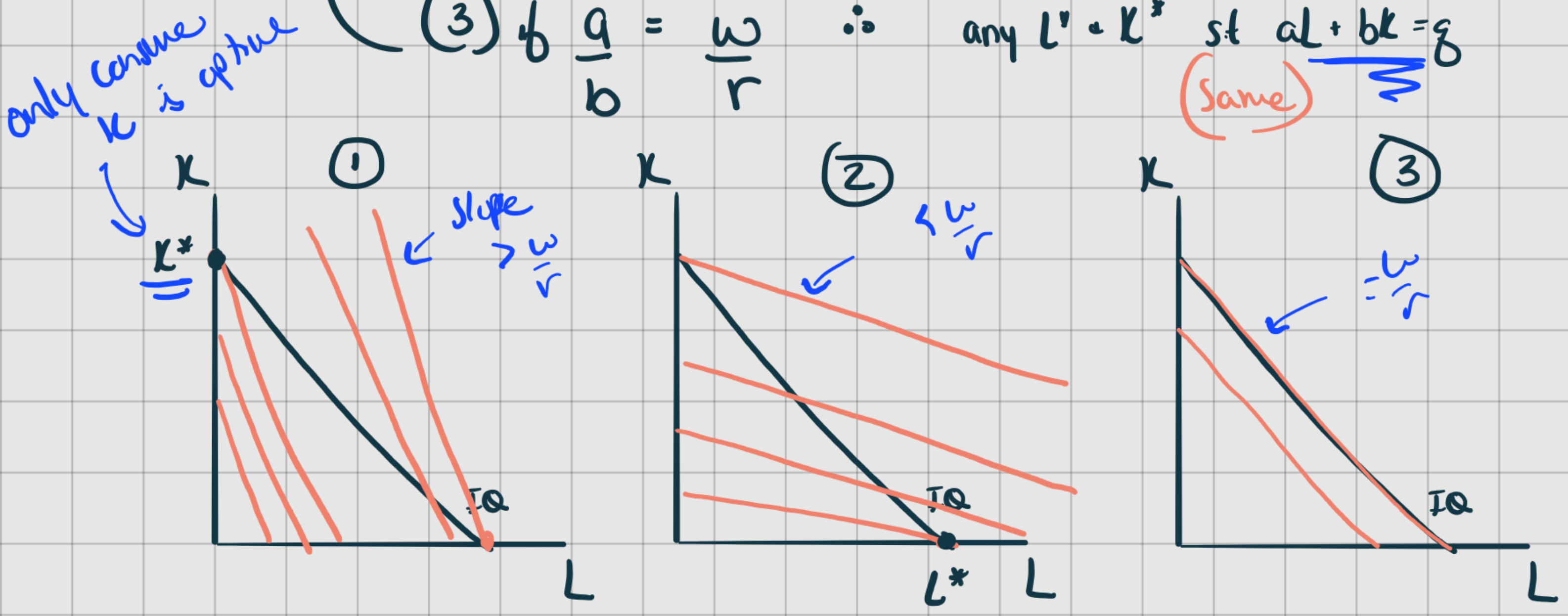
$\cancel{a}L + bK = q \quad \frac{a}{w} < \boxed{\frac{b}{r}}$

min cost

(1) if $\frac{a}{b}$ (MPL/MPK) $< \boxed{\frac{w}{r}}$ ∴ $L^* = 0$ & $q = bK \Rightarrow K^* = \frac{q}{b}$ (IC steeper)

(2) if $\frac{a}{b} \geq \frac{w}{r}$ ∴ $K^* = 0$ & $q = aL \Rightarrow L^* = \frac{q}{a}$ (IQ steeper)

(3) if $\frac{a}{b} = \frac{w}{r}$ ∴ any L^* & K^* st $aL + bK = q$ (Same)



↳ cost function: $wL + rK = C$

$$c(q,r,w) = \begin{cases} (1) \text{ if } \frac{a}{b} < \frac{w}{r} \quad (L^* = 0, K^* = \frac{q}{b}) \\ = \left[\Rightarrow C = rK^* = r\frac{q}{b}\right] \\ (2) \text{ if } \frac{a}{b} \geq \frac{w}{r} \quad (L^* = \frac{q}{a}, K^* = 0) \\ = \left[\Rightarrow C = wL^* = w\frac{q}{a}\right] \end{cases}$$

plugged in optimal value to find cost in terms of q, r, w

$\frac{q}{b}$ vs $\frac{w}{r}$ $\Rightarrow$ $\frac{r}{b}$ vs $\frac{w}{a}$

⇒ Simplify

$$c(q,w,r) = \left[q \ * \right] \begin{cases} (1) \ \boxed{\frac{r}{b}} & \text{if } \frac{r}{b} < \frac{w}{a} \quad K \text{ only} \\ (2) \ \boxed{\frac{w}{a}} & \text{if } \frac{r}{b} \geq \boxed{\frac{w}{a}} \quad L \text{ only} \end{cases}$$

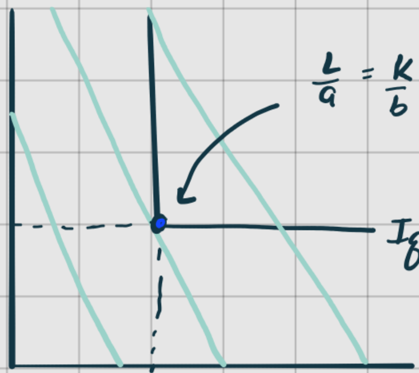
$$\left[= \boxed{q} * \min\left\{ \frac{w}{a}, \frac{r}{b} \right\} \right]$$

3) Kink sol^n

Compliments

$$q = f(K,L) = \min\left\{ \frac{L}{a}, \frac{K}{b} \right\}$$

found when Both equal



$$\min \ wL + rK$$
$$\text{st } \min\left\{ \frac{L}{a}, \frac{K}{b} \right\}$$

$$K = \frac{Lb}{a} \qquad \min\left\{ \frac{L}{a}, \frac{Lb}{ba} \right\}$$

$$\frac{L}{a} = \frac{K}{b} \quad \text{* optimality condition *}$$

↳ solve optimality into constraint

sub from optimality condition

$\frac{K}{b} = \frac{L}{a}$

constraint → $q = \min\left\{ \frac{L}{a}, \frac{L}{a} \right\}$

$$q = \frac{L}{a} \Rightarrow \therefore \boxed{L^* = qa}$$

← w/o optimality condition

$\therefore \frac{K}{b} = \frac{L}{a} \Rightarrow \frac{K}{b} = \frac{qa}{a} \Rightarrow \boxed{K^* = qb}$

Cost function:

$$c(q, w, r) = w\bar{L}^* + r\bar{K}^*$$
$$= wqa + rqb$$
$$\left[= q(wa + rb)\right]$$

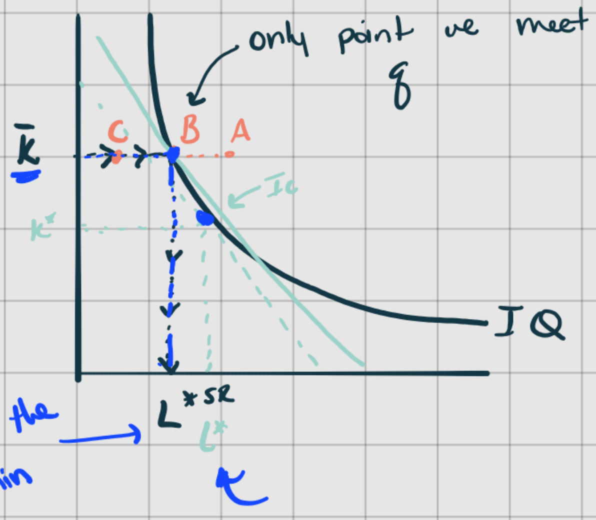
Sticky

SR conditional demand for L (or K) — conditional on $\bar{K}$

SR ∴ $K = \bar{K}$ ← can't change K

↓

$q = f(L, \bar{K})$



* b/c $\bar{K}$ isocost might not be min cost... there is cheaper * mix q L + K to produce q BUT can't Δ K!

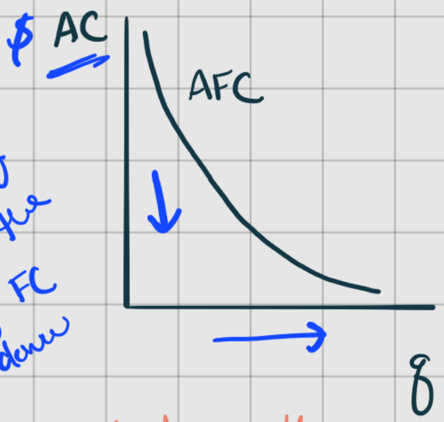
Cost curves:

(1) **Total cost**: $TC = C_v(q) + F$

- $C_v(q)$: variable cost (per unit)
- F: Fixed cost
  - **Fixed**: does not depend on q
  - **Quasi**: only if $q > 0$
  - Sunk: not recoverable

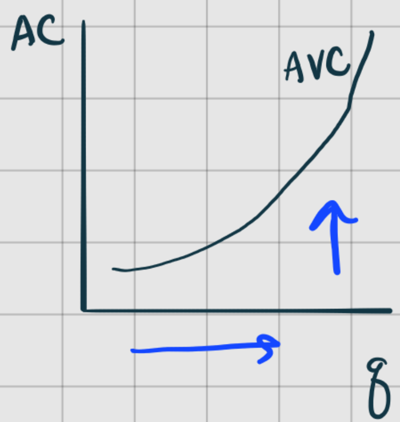
(2) **Average cost**: $AC = \frac{TC}{q} = \frac{C_v(q)}{q} + \frac{F}{q}$

$= \underline{AVC(q)} + \underline{AFC(q)}$

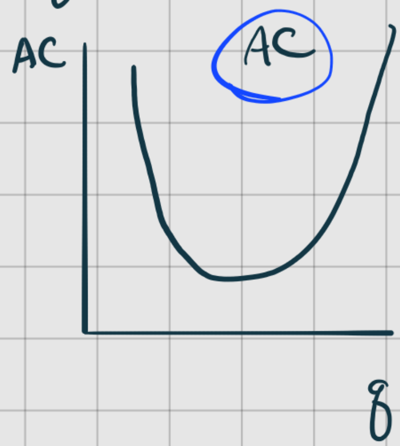


for every 1 in q the share of FC goes down

*gets smaller w/ more

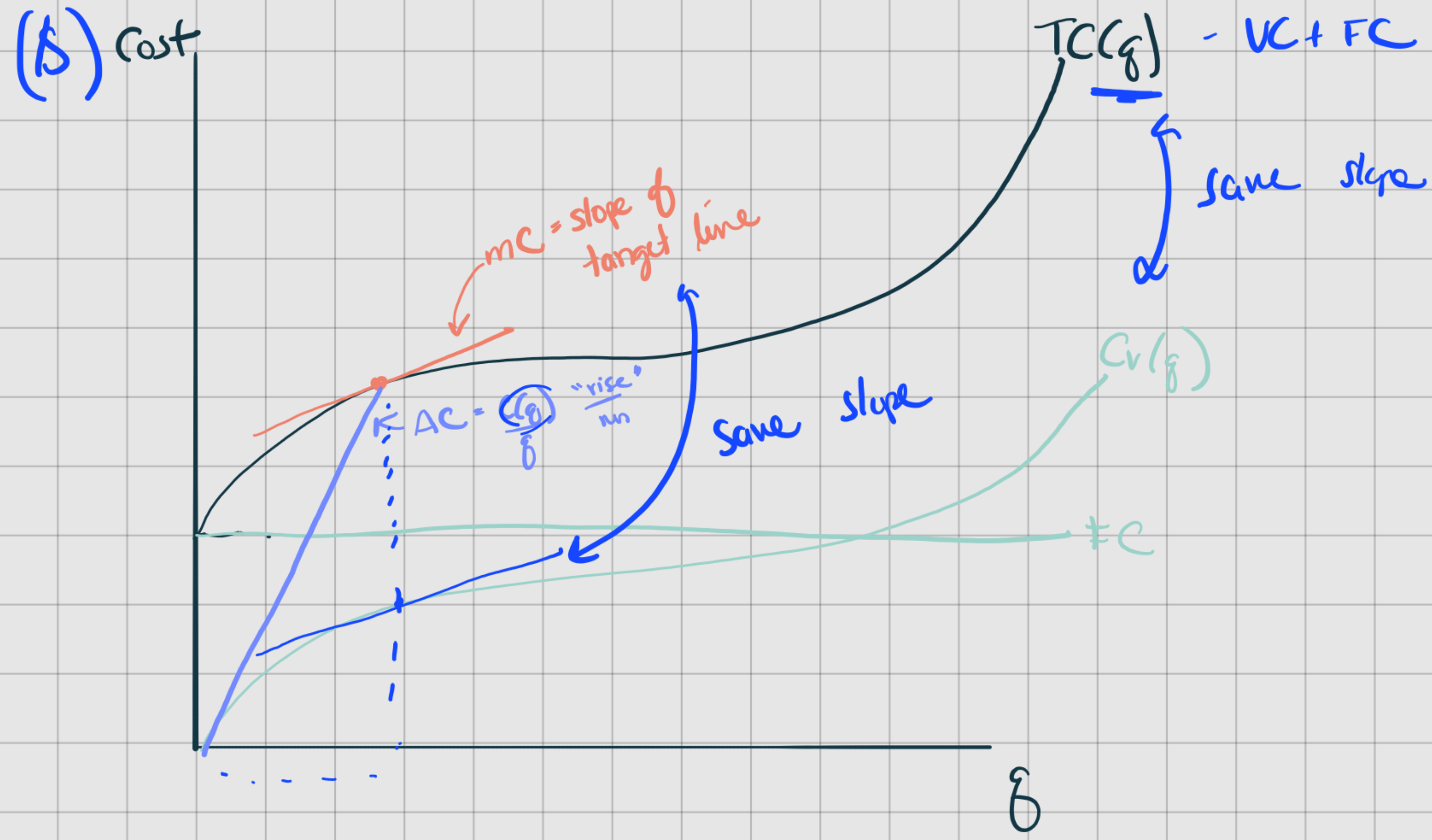


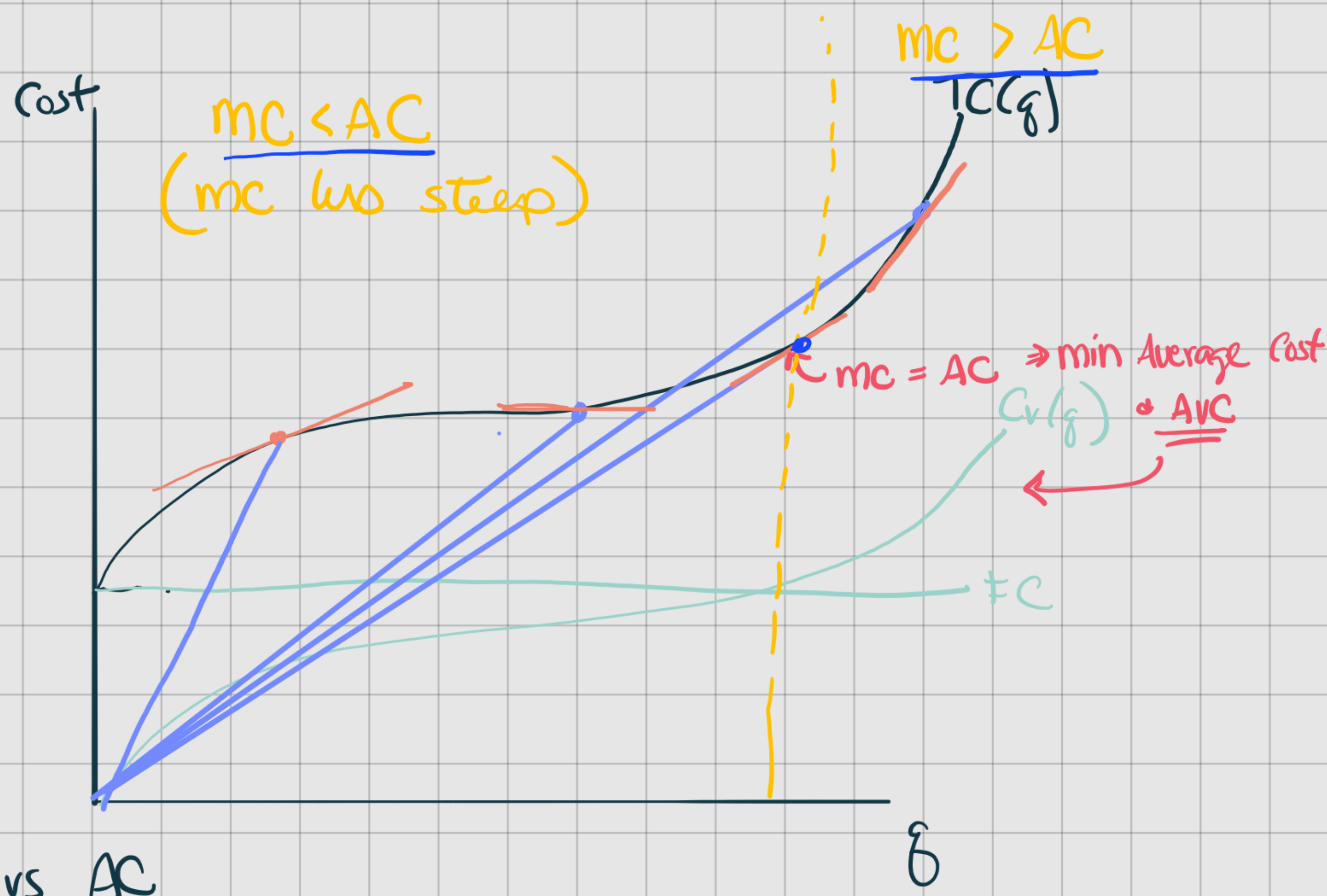
gets higher w/ more



(3) **Marginal cost**: Δ in cost due to change in q

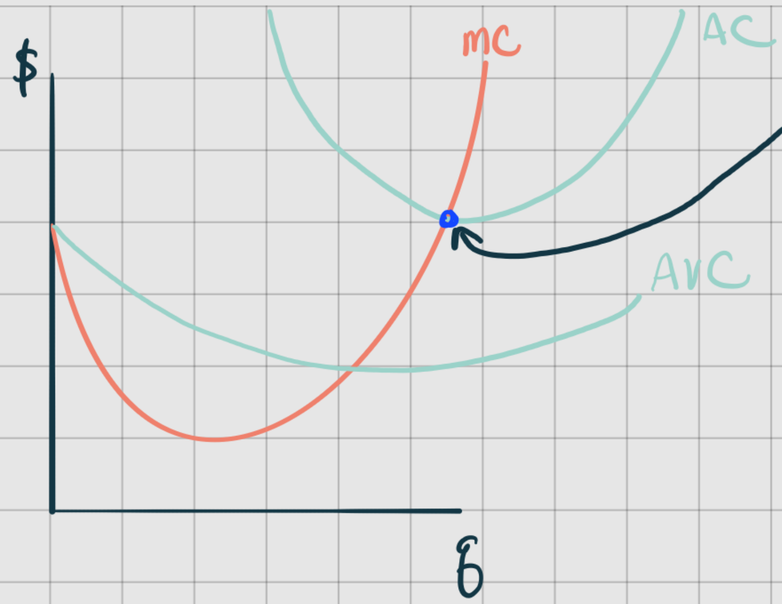
$$MC = \frac{\partial TC}{\partial q} = \frac{\partial C_v(q)}{\partial q} + \frac{\partial F}{\partial q} = \boxed{\frac{\partial C_v(q)}{\partial q}}$$





mc vs AC

- mc = AC @ q = 0 + optimal q*
- mc crosses @ min AC & AVC
  - mc < AC when AC ↓
  - mc > AC when AC ↑



[AC crosses MC @ min AC]

↑ similar to AP + MP

RTS & AC : $AC = \frac{TC(q)}{q} = \frac{wL^* + rK^*}{q}$

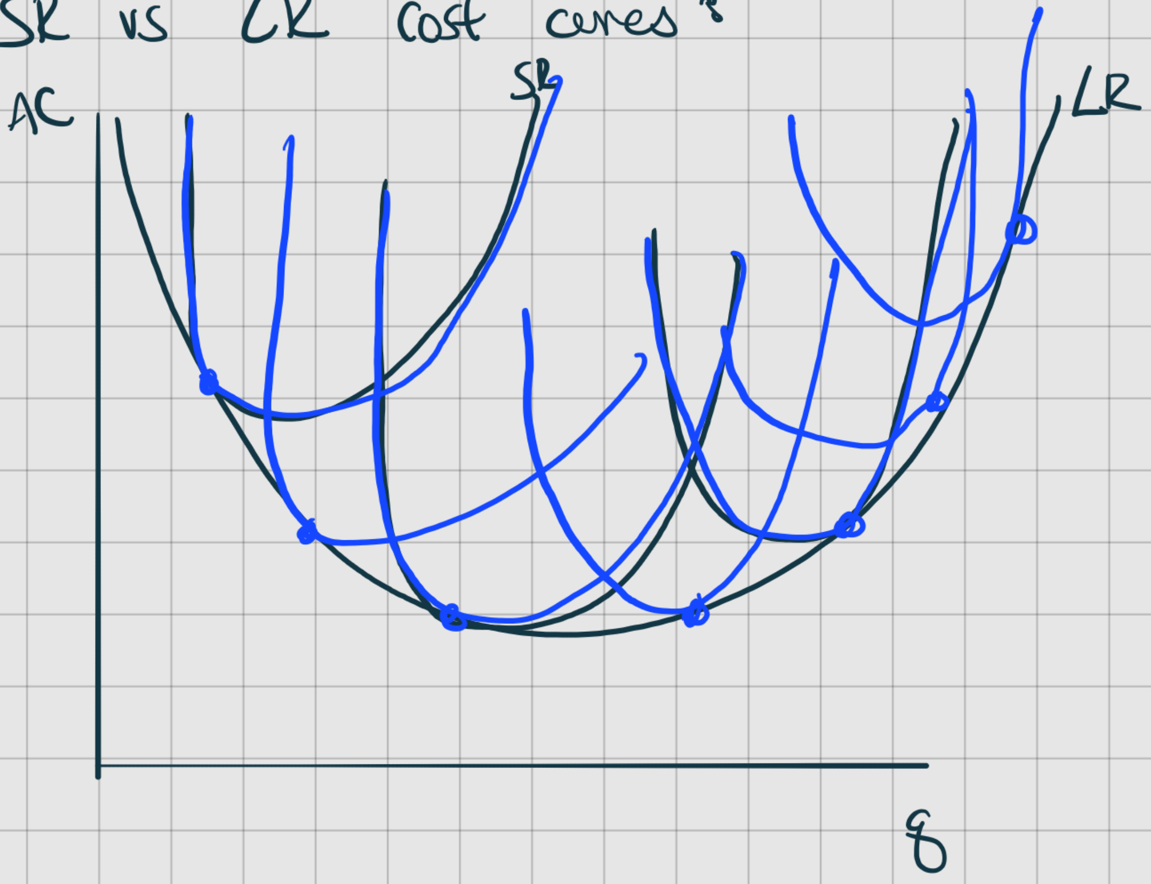
(1) ↑ q & AC ↓ ∴ Increasing RTS ← adding more for less

(2) ↑ q & AC DNC ∴ constant RTS

(3) ↑ q & AC ↑ ∴ decreasing RTS

↑ more expensive to add units.

SR vs LR cost curves:



* LR cost curves envelope SR cost curves

(Q1) min $wL + rK$ ← CD. Tangency

st $q = 3L^{1/2}K^{1/2}$

(1) $\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{3K^{1/2}\frac{1}{2}L^{-\frac{1}{2}}}{3L^{1/2}\frac{1}{2}K^{1/2}} = \frac{w}{r}$

$$\boxed{\frac{K}{L} = \frac{w}{r}} \Rightarrow K = \frac{w}{r}L$$

Sub into constraint

$3L^{1/2}\left(\frac{w}{r}L\right)^{1/2} = q$

$3L^{1/2 \cdot 1/2}\left(\frac{w}{r}\right)^{1/2} = q$

$\boxed{L^*} = \frac{q}{3}\left(\frac{r}{w}\right)^{1/2}$

$L = \frac{q}{3}\left(\frac{r}{w}\right)^{1/2}$

$K = \frac{w}{r}\left(\frac{q}{3}\right)\left(\frac{r}{w}\right)^{1/2}$

$w^{1-\frac{1}{2}}$

$$\left[K^* = \frac{q}{3}\left(\frac{w}{r}\right)^{1/2}\right]$$

$C = w\,\frac{q}{3}\left(\frac{r}{w}\right)^{1/2} + r\left(\frac{q}{3}\left(\frac{w}{r}\right)^{1/2}\right)$

(2) $q = 90$, $w = 25$, $r = 9$ > cost = ?

$L^* = 18$

$K^* = 50$

cost = $wL + rK = 25(18) + 9(50) = \$900$

(3) SR

$Q = 84$, $w = 2$, $r = 3$, $\bar{K} = 16$ > cost = ?

$Q = 3L^{1/2}\bar{K}^{1/2}$ ← 1 unknown, 1 equation

$L = \left(\frac{Q}{3}\right)^2\frac{1}{K} = \frac{Q^2}{9}\frac{1}{16} = \frac{84^2}{144} = \boxed{49}$

$C = 2(49) + 3(16) = \$146$ (L^{*SR}, $\bar{K}$)

only solve for L b/c given $\bar{K}$

(Q3) min $wL + rK$

st $Q = 2L + K$ ← Sub

(1) LR L^* & K^*

Better deal

$\frac{MP_L}{w}$ vs $\frac{MP_K}{r}$ → $\frac{2}{w}$ vs $\frac{1}{r}$

① if $\frac{2}{w} > \frac{1}{r}$ ∴ $K^* = 0$ & $Q = 2L^*$ $L^* = Q/2$

② if $\frac{2}{w} < \frac{1}{r}$ ∴ $L^* = 0$ & $Q = K^*$

③ if $\frac{2}{w} = \frac{1}{r}$ ∴ L^* & K^* any that meet Q

(2) $Q = 20$
$w = 6$
$r = 1$

$\frac{2}{w}$ vs $\frac{1}{r}$

$\frac{2}{6} < \frac{1}{1}$ ∴ $L^* = 0$
$K^* = Q = 20$

K is a better deal

$C = wL + rK$
$= 6(0) + 1(20)$
$= \$20$

(Q6) AC, AVC, AFC & MC

only VC	FC, VC	VC	FC, VC
$TC = 5Q$	$TC = 120 + 6Q$	$TC = 6Q^2$	$TC = 140 + 5Q^2$
$AC = \frac{5Q}{Q} = 5$	$AC = \frac{120}{Q} + 6$	$AC = 6Q$	$AC = \frac{140}{Q} + 5Q$
$AFC = 0$	$AFC = \frac{120}{Q}$	$AFC = 0$	$AFC = \frac{140}{Q}$
$AVC = \frac{5Q}{Q} = 5$	$AVC = 6$	$AVC = 6Q$	$AVC = 5Q$
$MC = 5$	$MC = 6$	$MC = 12Q$ (≠ *)	$MC = 10Q$ (≠ *)