Game Theory

- Game: interaction w strategic interdependence

types
- timing
  - Static
  - Dynamic (sequential)
- information
  - complete (payoff functions common knowledge)
  - Perfect (know full game history)

Simultaneous -
- Complete info of payoff
- imperfect info about rival

① Static, simultaneous move games (one shot)

players · actions · comparisons

	P2 ←	P2 →
P1 ↑	2, 1	0, 0
P1 ↓	0, 0	1, 2

Best move given the action of the other player

P1: P2 ← = ↑ $\quad BR_A(\leftarrow) = \uparrow$
P1: P2 → = ↓ $\quad BR_A(\rightarrow) = \downarrow$

P2: P1 ↑ = ← $\quad BR_B(\uparrow) = \leftarrow$
P2: P1 ↓ = → $\quad BR_B(\downarrow) = \rightarrow$

if payoffs equal can be set $BR_B(\downarrow) = \{\leftarrow, \rightarrow\}$
↳ ∴ player is indifferent to choice

No Dominant strategy

↳ Best Response: Choice player will make to max payoff given others player strategy

* Can hae more than one ∴ set of optimal *

↳ **Dominant strategy**: Strategy produces higher payoffs regardless of what rival does

Best choice all they time

⇒ DS eq^{m}: combination of strats that players in game arrive at when everyone has own DS

* all players need DS *

always goes left

P2

always goes up

P1

	←	→
↑	7, 4	3, 3
↓	6, 5	1, 4

P1 DS: ↑

P2 DS: ←

DSE: P1 plays ↑

P2 plays ←

↳ **Nash eq^m**: mutually responding to other strategies & no player has an $\boxed{\text{incentive to deviate}}$

what type of market structure?

Cournot

↑ individual firms max π separately

P2

DS of player 2

P1

	←	→
↑	3 5	5 6
↓	2 1	7 2

[NE: (↓, →)]

Not a dominant strategy eq^m b/c only P2 has a DS

* DSE are NE *

NE not always DSE

→ Important Games:

(a) coordination game

A \ B	L	R
T	2, 1	0, 0
B	0, 0	1, 2

each has different moves that depend on others moves.

players need to coordinate to make sure high payoff

(b) cooperation game
(social / prisoners dilemma)

A \ B	C	D
C	5, 5	1, 7
D	7, 1	2, 2

DSE: (D, D)

↳ but (C, C) has higher payoff!

(c) chicken

A \ B	Sw	St
Sw	0, 0	-1, 1
St	1, -1	-2, -2

NE: (St, Sw), (Sw, St)

↳ one person chickens out

(d) Matching Pennies

(A) \ (B)	H	T
H	$\underline{1}$, -1	-1, $\overline{1}$
T	-1, $\overline{1}$	$\underline{1}$, -1

No NE in Pure Strat!

∴ mixed strategy

we don't know the rate that players take each action

Pure strategy : strategy played w/ certainty $(P = 1)$

mixed strategy : strategy where actions are random $(P \neq 1)$

⇒ more than 1 strategy w/ prob > 0

↳ Mixed strategy

(A) \ (B)		q L	$(1-q)$ R
p	T	$\underline{2}$, $\overline{1}$	0, 0
$(1-p)$	B	0, 0	$\underline{1}$, $\overline{2}$

$q \cdot p$

$(1-p)(1-q)$

want EU to be (+)

use player A payoffs

prob of being in that cell

$$EU_A(p|q) : p \cdot q \cdot \underset{\text{payoff}}{2} + (1-q)(\cancel{p})(0) + (1-p)(\cancel{q})(0) + (1-p)(1-q)1$$

$$= p(2q - (1-q)) + (1-q)$$

$$= p(3q - 1) + (1-q)$$

all of this doesn't matter

$(3q - 1)$

$q = 1/3$

depend on P2

$$BR_A : p^* = \begin{cases} 1 & \text{if } q > 1/3 \rightarrow \text{P1 plays T} \\ 0 & \text{if } q < 1/3 \rightarrow \text{P1 plays B} \\ [0,1] & \text{if } q = 1/3 \rightarrow \text{either P1 indifferent} \end{cases}$$

expected payoff of P2 | prob of P1

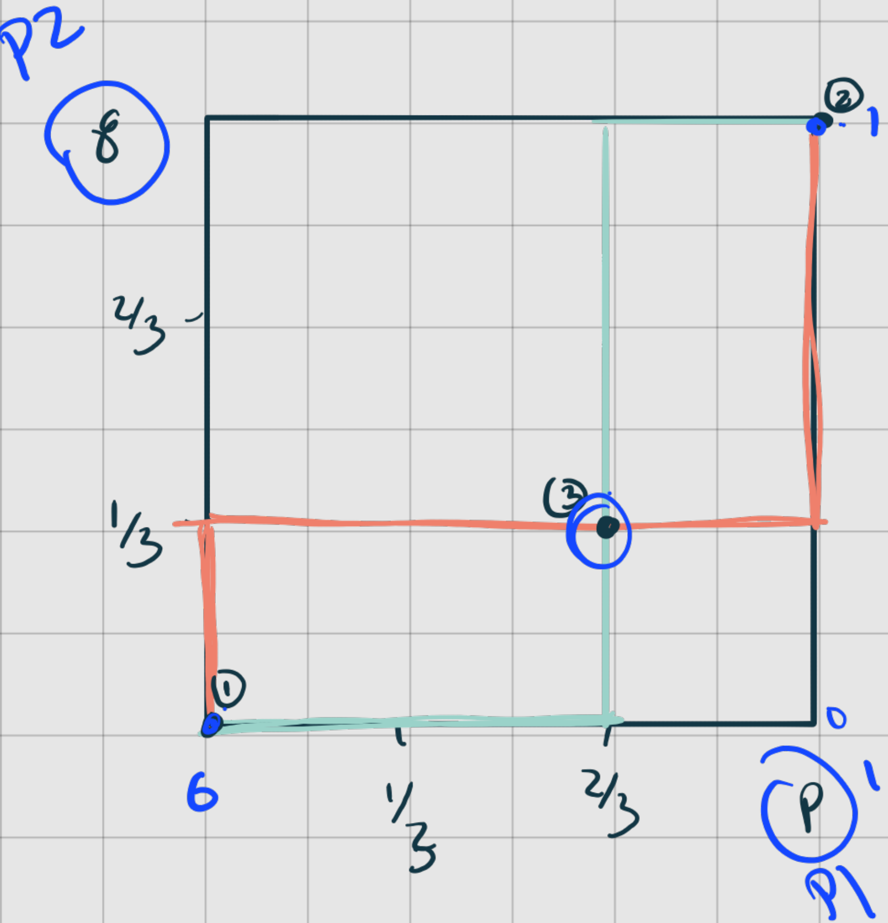
$$EU_B(q|p) = p \cdot q \cdot 1 + (1-q)(p)(0) + (1-p)(q)(0) + (1-p)(1-q)2$$

$$= q(3p-2) + 2(1-p) \rightarrow p = 2/3$$

$$BR_B : q^* = \begin{cases} 1 & \text{if } p > 2/3 \\ 0 & \text{if } p < 2/3 \\ [0,1] & \text{if } p = 2/3 \end{cases}$$

P1

$3p-2$
if $p=0$: -2 if $=1$: 1



PS NE

Eq ①: $p=0$, $q=0 \Rightarrow (B,R)$

Eq ②: $p=1$, $q=2 \Rightarrow (T,L)$

Eq ③: $p=2/3$, $q=1/3$) $\left(\frac{2}{3}, \frac{1}{3}\right)$

ms NE

P1 plays T w/ prob 2/3

P2 plays L w/ prob 1/3

⇒ Calculating ms NE:

		q	1-q
		(B) H	T
P	H	1, -1	-1, 1
(A) 1-P	T	-1, 1	1, -1

No PS NE

$$\underline{EU_A(p|q)} = pq(1) + (1-p)q(-1) + p(1-q)(-1) + (1-p)(1-q)(1)$$

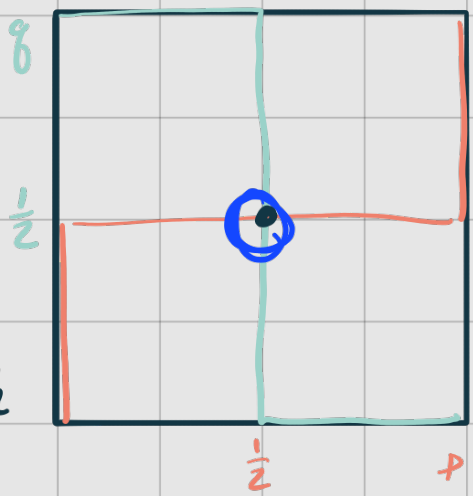
get all p together

Collected all like terms that have P

$$= p(q - 1 + q - 1 + q + q) + [\text{does not depend on } p]$$

$$= p\,2\underbrace{(2q+1)}_{(+)} + [\text{Something}]$$

$$BR_A : p^*_a = \begin{cases} 1 & \text{if } q > 1/2 \rightarrow \text{P1 plays H} \\ 0 & \text{if } q < 1/2 \rightarrow \text{P1 plays T} \\ \boxed{[0,1] \quad \text{if } q = 1/2} \end{cases}$$

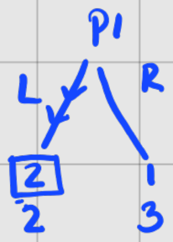


BY SYM

$$BR_B : q^*_A = \begin{cases} 1 & \text{if } p > 1/2 \\ 0 & \text{if } p < 1/2 \\ [0,1] & \text{if } p = 1/2 \end{cases}$$

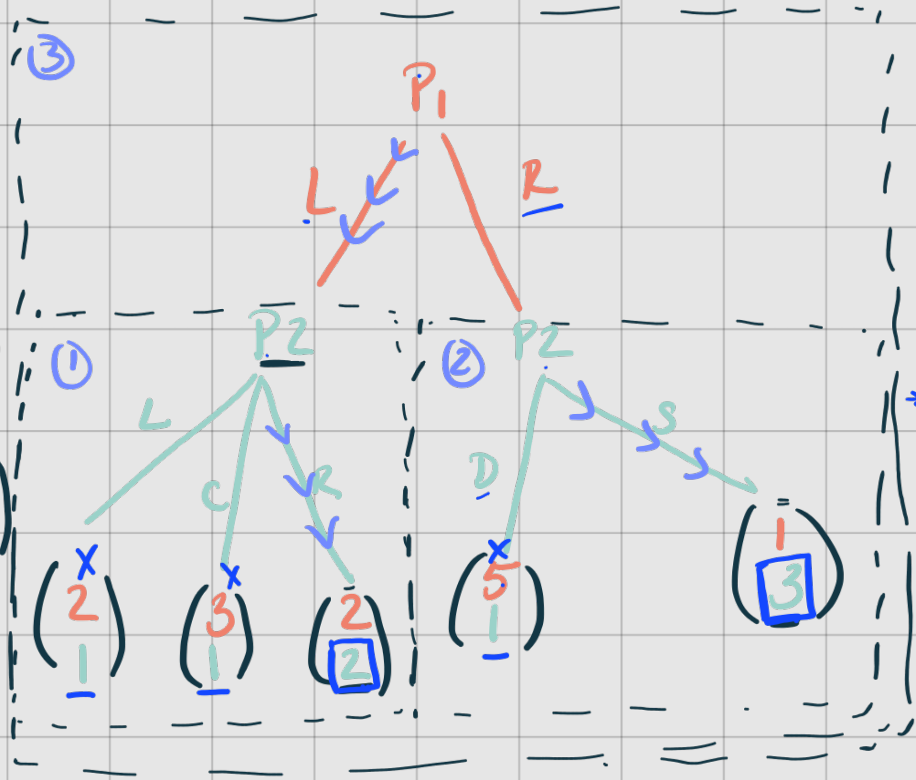
$$\Rightarrow \left[\text{MS NE} : \begin{array}{l} p = 1/2 \\ q = 1/2 \end{array} \right]$$

② Sequential Dynamic games - leader follower



3 Sub game

(node & analyzed in isolation)



→ P1 strategy : (L)

→ P2 strategy : (R, S)

(node 1, node 2, ...)

P1 P2

(L ; R, S)

↑

strategy profile

action : move @ each node

Strategy : complete plan for actions @ all nodes

Strategy profile : combination of strategies by all player!

P1 : L + P2 : L, d

↳ Sub game Perfect Eqm : A SP that constitutes a NE for every proper subgame

(aka backwards induction)

① P2 → R

② P2 → S

→ ③ P1 → L

SG NE : P1 = L   P2 = (R, S)

Homework # 9 (2,3,5)

Question #2

		P2 Nadal	
		DL q	CC $(1-q)$
P1 Federer	DL p	$(5,\underline{5})$	$(\underline{8},2)$
	CC $(1-p)$	$(\underline{9},1)$	$(2,\underline{8})$

prob of winning using
DL - down the line
CC - crosscourt

1) Find Pure Strategy NE

Pure Strategy ⇒ compare across

↳ how to solve

if N Play DL ... then F plays CC
CC ... then F Plays DC

→ Not the same No PS

⇒ No PS

2) Find $\boxed{\text{MS NE}}$

player 1 use payoffs of P2

↳ solve for P (Federer)

Solve for the value p

p that equates N payoffs from positioning for F DL or CC

payoff of P2 + prob of P1

DL CC

$$5p + 1(1-p) = 2p + 8(1-p)$$

← when they are indifferent btw the 2 P2

↓

Nadal Plays DL & experiences payoff of 5 w/ prob p (given p is prob Feberer plays DL...)

		DL q	CC $(1-q)$
P1	DL p	$(5,\underline{5})$	$(\underline{8},2)$
	CC $(1-p)$	$(\underline{9},1)$	$(2,\underline{8})$

$$5p + 1 - p = 2p + 8 - 8p$$
$$4p + 1 = 8 - 6p$$
$$10p = 7$$

$$\boxed{p = \frac{7}{10}}$$

when is P1 b/w indifferent the choice of p2

↳ Solve for q (Nadal)

DL CC

payoffs of P1 + prob of p2

$$5q + 8(1-q) = 9q + 2(1-q)$$
$$5q + 8 - 8q = 9q + 2 - 2q$$
$$8 - 3q = 7q + 2$$
$$6 = 10q$$
$$q = \boxed{6/10}$$

Nadal plays DL w̄ prob 6/10 ✓
CC w̄ prob 4/10

Federer plays DL w̄ prob 7/10
CC w̄ prob 3/10

3) Winning probability

Nadal: if DL ⇒ $5\left(\frac{7}{10}\right) + \frac{3}{10} = 3.8$
if CC ⇒ $2\left(\frac{7}{10}\right) + 8\left(\frac{3}{10}\right) = 3.8$

Federer: if DL ⇒ $5\left(\frac{6}{10}\right) + 8\left(\frac{4}{10}\right) = 6.2$
CC ⇒ $9\left(\frac{6}{10}\right) + 2\left(\frac{4}{10}\right) = 6.2$

* payoff is prob of winning in this question.

0.38 prob for N + 0.62 for F.

Question #3

		NK Build	NK Refrain
SK	Build	$(\boxed{6},\boxed{6})$	(5,1)
	Refrain	(3,8)	$(\boxed{12},\boxed{12})$

1) dominate strate (aka play all the time)
No dominate.

2) Pure strategy NE:

		NK Build n	NK Refrain $1-n$
SK	Build s	(6,6)	(5,1)
	Refrain $1-s$	(3,8)	(12,12)

(B,B) + (R,R)

3) Response functions $b_s(n)$ $b_n(s)$

best response of SK given actions of NK

NK: Payoff

$6 \cdot s \cdot n + 8(1-s)n + (1-n)s + 12(1-n)(1-s)$

$6sn + 8n - 8sn + s - sn + 12 - 12n - 12s + 12sn$

$9sn - 11s - 4n + 12$

$\underbrace{n(9s-4)}_{s = 4/9} + \underbrace{12 - 11s}_{\text{do not matter for N}} \Rightarrow s = \frac{4}{9}$ (depending on what s is relative will change BR

$$b_n(s) = \begin{cases} 0 & \text{if } s < \frac{4}{9} \\ \in [0,1] & \text{if } s = \frac{4}{9} \\ 1 & \text{if } s > \frac{4}{9} \end{cases} \qquad n=$$

$$n\left(9\left(\tfrac{5}{9}\right)-4\right)+12-11\left(\tfrac{5}{9}\right)$$
$$= n(1) + 12 - \tfrac{55}{9}$$
$$= 5.9$$

* # larger if $n>0$ *

$$n\left(9\left(\tfrac{4}{9}\right)-4\right)+12-11\left(\tfrac{4}{9}\right)$$
$$= n(0) + 7.11$$

* # does not change *

$$n\left(9\left(\tfrac{3}{9}\right)-4\right)+12-11\left(\tfrac{3}{9}\right)$$
$$= n(-1) + 8.33$$

* # smaller if $n=1$ *

SC Payoff:

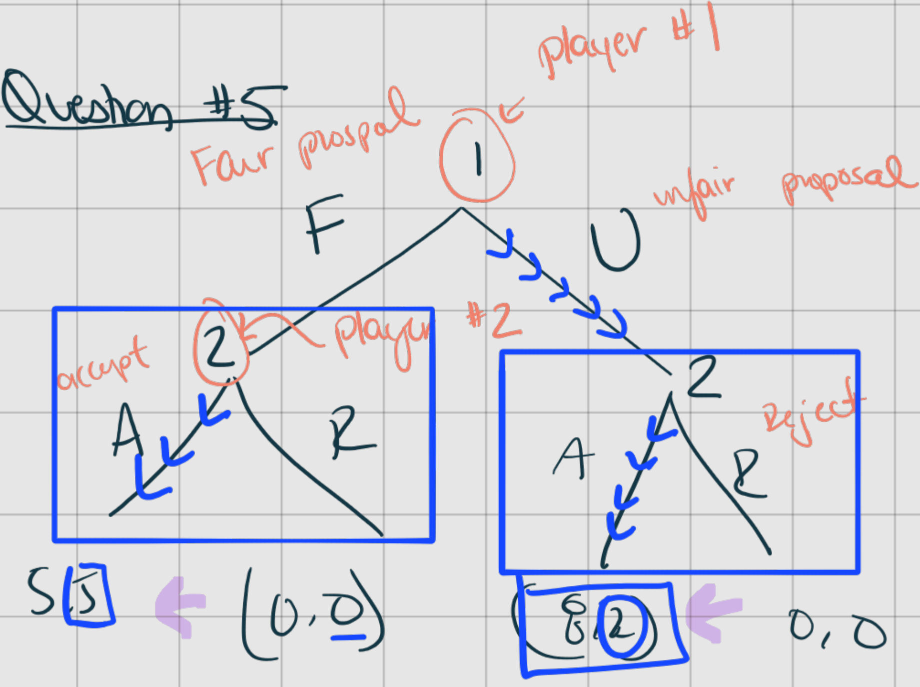
$$6sn + 5(1-n)s + 3(1-s)n + 12(1-n)(1-s)$$
$$6sn + 5s - 5ns + 3n - 3sn + 12 - 12n - 12s + 12sn$$
$$10sn - 7s - 9n + 12$$
$$s(10n-7) - 9n + 12 \quad (n = 7/10)$$

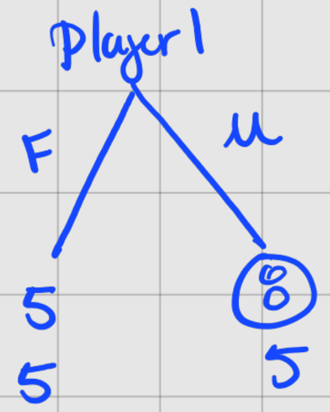
$$b_s(n) = \begin{cases} 0 & \text{if } n < 7/10 \\ \in [0,1] & n = 7/10 \\ 1 & n > 7/10 \end{cases}$$

4) ms NE

$n = 7/10$ & $s = 4/9$ is ms NE

Question #5



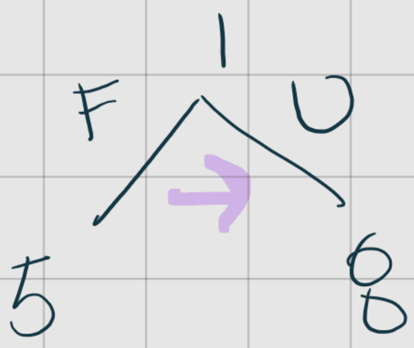


P1 = U
P2 = (A, A)

(U; A, A)

1) Find subgame perfect NE .

→ compare starting from bottom.



∴ Player 1 unfair

Player 2 Accept & Accept

$$\left(u \,;\, A , A \right)$$

* need to specify every note